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LETTER TO THE EDITOR

On coherent spin states

R Holtz† and J Hanus‡

†Centre de Physique Théorique, CNRS, 31 Chemin J Aiguier, 13 Marseille (9e), France

‡Groupe de Physique des Etats Condensés, équipe associée au CNRS, Département de Physique, UER de Luminy, 70 route Léon Lachamp, 13288 Marseille, Cédex 2, France

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Abstract. Some properties of so called coherent spin states, analogous to the coherent states of the harmonic oscillator, are discussed. Polynomials of order $2S$ play an important role. The representation of spin operators in the space of coherent spin states is found. This representation is used to study the rotation operator.

Coherent states have been particularly convenient to study bosons problems. For spin hamiltonians, such as the ferromagnetic Heisenberg hamiltonian, one often needs to transform the spin operators to recover boson commutation properties. These Holstein–Primakoff and Dyson–Maleev transformations introduce complications into the hamiltonians. One might hope to avoid such modifications by the use of the coherent spin states first introduced by Radcliffe (1971), since they are directly defined with spin operators. In this letter we stress the role of polynomials of order $2S$ in the theory of coherent spin states (CSS). We also study the rotation operator in CSS space.

Let us recall the definition and some properties of CSS given by Radcliffe. For a particle of spin S, where the state $|S,m\rangle$ is an eigenstate of the operators S^2 and S_z, it is convenient to introduce the number of spin deviations $p = S-m$ and the operator 'number of deviation' Δ such that

$$\Delta|S,p\rangle = p|S,p\rangle.$$

The coherent spin state $|\mu\rangle$ (a short-hand notation for $|S,\mu\rangle$) is defined by

$$|\mu\rangle = \exp(\mu S^-)|S,0\rangle = \sum_{\mu=0}^{2S} \binom{2S}{p}^{1/2} \mu^p|S,p\rangle$$

where $\mu \in \mathbb{C}$.

Unlike Radcliffe we consider here unnormalized $|\mu\rangle$ states. Two CSS are not orthogonal and their overlap is

$$\langle\lambda|\mu\rangle = (1+\lambda^*\mu)^{2S}. \tag{1}$$

Nevertheless these states form an overcomplete basis

$$\int_{\mathbb{C}} |\mu\rangle\langle\mu|\mathrm{d}M(\mu) = \sum_{p=0}^{2S} |S,p\rangle\langle p,S| = \mathbf{1} \tag{2}$$

with a weight factor

$$\mathrm{d}M(\mu) = \frac{2S+1}{\pi}(1+|\mu|^2)^{-2S-2}\mathrm{d}^2\mu.$$

The introduction of CSS replaces discrete summation by an integration at the price of overcompleteness. The overlap between $|p\rangle$ (for $|S,p\rangle$) and $|\mu\rangle$ is given by:

$$\langle p|\mu\rangle = \binom{2S}{p}^{1/2} \mu^p. \tag{3}$$

With the three basic relations (1), (2) and (3) we can now consider the role of order $2S$ polynomials in CSS formalism.

A general state $|f\rangle$ of a particle of spin S is given by

$$|f\rangle = \sum_{p=0}^{2S} C_p|p\rangle$$

in the spin deviation space. If we write $f(\mu^*) = \langle\mu|f\rangle$, then $f(\mu^*)$ is a polynomial of order $2S$ in μ^*:

$$f(\mu^*) = \sum_{p=0}^{2S} C_p\binom{2S}{p}^{1/2} \mu^{*p}$$

and the expansion of $|f\rangle$ in terms of CSS is

$$|f\rangle = \int_{\mathbb{C}} |\mu\rangle f(\mu^*)\mathrm{d}M(\mu).$$

From the definition of $|\mu\rangle$ one gets

$$S^-|\mu\rangle = \frac{\mathrm{d}}{\mathrm{d}\mu}|\mu\rangle$$

and one easily verifies that

$$\langle g|S^-|f\rangle = \int_{\mathbb{C}} [g(\lambda^*)]^*\left(-\lambda^{*2}\frac{\mathrm{d}}{\mathrm{d}\lambda^*}+2S\lambda^*\right)f(\lambda^*)\mathrm{d}M(\lambda)$$

$$\langle g|S^+|f\rangle = \int_{\mathbb{C}} [g(\lambda^*)]^*\left(\frac{\mathrm{d}}{\mathrm{d}\lambda^*}\right)f(\lambda^*)\mathrm{d}M(\lambda)$$

$$\langle g|\Delta|f\rangle = \int_{\mathbb{C}} [g(\lambda^*)]^*\left(\lambda^*\frac{\mathrm{d}}{\mathrm{d}\lambda^*}\right)f(\lambda^*)\mathrm{d}M(\lambda)$$

and in particular,

$$\langle\lambda|S^-|f\rangle = \left(-\lambda^{*2}\frac{\mathrm{d}}{\mathrm{d}\lambda^*}+2S\lambda^*\right)f(\lambda^*) = S_{\lambda^*}^{\;-}f(\lambda^*) \tag{4a}$$

$$\langle\lambda|S^+|f\rangle = \frac{\mathrm{d}}{\mathrm{d}\lambda^*}f(\lambda^*) = S_{\lambda^*}^{\;+}f(\lambda^*) \tag{4b}$$

$$\langle\lambda|\Delta|f\rangle = \lambda^*\frac{\mathrm{d}}{\mathrm{d}\lambda^*}f(\lambda^*) = \Delta_{\lambda^*}(\lambda^*). \tag{4c}$$

One gets the matrix elements of the spin operators between two coherent states

$$\langle\lambda|(S^-)^n|\mu\rangle = (S_{\lambda^*}{}^-)^n(1+\lambda^*\mu)^{2S}$$
$$\langle\lambda|(S^+)^n|\mu\rangle = (S_{\lambda^*}{}^+)^n(1+\lambda^*\mu)^{2S}$$
$$\langle\lambda|\Delta^n|\mu\rangle = \Delta_{\lambda^*}{}^n(1+\lambda^*\mu)^{2S}.$$

These relations show that the utilization of CSS formalism leads to operations on polynomials of order $2S$. These representations of the operators S^+, S^- and Δ now will be used to give matrix elements for a rotation operator R.

If α, β and γ are the Euler angles, the expression of a rotation $R(\alpha,\beta,\gamma)$ in terms of spin operators is

$$R(\alpha,\beta,\gamma) = \exp(-\mathrm{i}\alpha S_z)\exp(-\mathrm{i}\beta S_y)\exp(-\mathrm{i}\gamma S_z).$$

A state $|\phi\rangle$ of a particle of spin S, on which the rotation $R(\alpha,\beta,\gamma)$ is acting, has a component on a state $|\lambda\rangle$ given by $\langle\lambda|R(\alpha,\beta,\gamma)|\phi\rangle$.

Using (4) we find

$$\begin{aligned}\langle\lambda|R(\alpha,\beta,\gamma)|\phi\rangle &\\ &= \exp[-\mathrm{i}S(\alpha+\gamma)]\exp\left(\mathrm{i}\alpha\lambda^*\frac{\mathrm{d}}{\mathrm{d}\lambda^*}\right)\exp\left(-\frac{\beta}{2}(\lambda^{*2}+1)\frac{\mathrm{d}}{\mathrm{d}\lambda^*}+\beta S\lambda^*\right)\\ &\quad\times\exp\left(\mathrm{i}\gamma\lambda^*\frac{\mathrm{d}}{\mathrm{d}\lambda^*}\right)\phi(\lambda^*) = R_{\lambda^*}(\alpha,\beta,\gamma)\phi(\lambda^*).\end{aligned}$$

All these operations can be performed, using the well known relation

$$\exp\left(a\frac{\mathrm{d}}{\mathrm{d}\lambda^*}\right)f(\lambda^*) = f(\lambda^*+a)$$

and one finds

$$R_{\lambda^*}(\alpha,\beta,\gamma)\phi(\lambda^*) = (a\lambda^*+b^*)^{2S}\phi\left(\frac{b\lambda^*-a^*}{a\lambda^*+b^*}\right),$$

where

$$a = \sin\tfrac{1}{2}\beta\exp[\tfrac{1}{2}\mathrm{i}(\alpha-\gamma)]$$

and

$$b = \cos\tfrac{1}{2}\beta\exp[\tfrac{1}{2}\mathrm{i}(\alpha+\gamma)].$$

We have here the standard parameters of the matrices of the SU(2) representation. An abstract group theoretical approach unifying boson coherent states and coherent spin states is given in the publication of Perelomov (1972).

Between two coherent spin states one gets

$$\langle\lambda|R(\alpha,\beta,\gamma)|\mu\rangle = R_{\lambda^*}(\alpha,\beta,\gamma)(1+\lambda^*\mu)^{2S} = (a\lambda^*+b^*+b\lambda^*\mu-a^*\mu)^{2S}.$$

Another application of the CSS theory can be found in Bellissard and Holtz (1973) who have studied the coupling of angular momenta in CSS space. Two recent works apply the same formalism: (i) Arecchi *et al* (1972) introduced CSS to tackle the coupling of atomic systems with the electromagnetic field in quantum optics; (ii) Lieb (1973) gave bounds to the quantum free energy of a spin system with the help of CSS theory.

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